A PROOF OF BRST INVARIANCE

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Abstract

Introducing a geometric normal ordering, we give a proof of BRST invariance of the states associated to an arbitrary genus Riemann surface with a puncture, in the operator formalism.

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1 Introduction

In the last years the interest in form
genus Riemann surfaces has been th
interest is motivated mainly by the n
tering amplitudes.

The operator formalism ([1],[2],[3])
the operator formalism, we associate
tured Riemann surface on which we l
state is defined as the state annihilate
In the cases we are interested, there
and this state is thus completely det
ubov transformation of the ordinary v
to invariances of the path integral un
nihilation operators in relation to the
are several fields in the theory, then w
associated to each field. In string t
the states associated to each X^μ (ma
conformal ghost fields b and c.

This formalism has many advanta
proved in terms of the sewing of surf
([1]), by making use of well-known re
three punctured spheres). They are s

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larly there is no such a proof for the BRST invariance of the states associated in this formalism to arbitrary genus punctured Riemann surfaces.

In this paper that proof is given for one punctured Riemann surfaces (bosonic string). The generalization to more punctures and super-Riemann surfaces (fermionic string) seems straightforward. In the last section we will make some comments on these subjects.

We first introduce in section 2 dual bases of vectors and quadratic differentials in a local coordinate around the puncture. These bases contain information about the global properties of the Riemann surface: they include the Teichmüller deformations and the vectors and quadratic differentials which are meromorphic at the puncture but extend holomorphically to the rest of the surface. The string path integral is invariant under shifts of b and c by precisely the latter elements of the bases. The associated conserved charges are the b and c Fourier modes corresponding also to them, and there are annihilation operators with respect to the Bogoliubov-transformed ghost vacuum $|\chi>$.

Using Serre duality and the Riemann-Roch theorem it is possible to find such bases (see appendix B of ref. [1]).

Serre duality again allows us to introduce, in section 3, a geometric normal ordering which is consistent with the substraction of vacuum expectation values on the surface.

In section 4, we briefly study the algebra of the geometrically normal ordered energy-momentum tensor Fourier modes, and the conformal anomaly cancellation.

The geometrically normal ordered BRST charge, introduced in section 5, trivially annihilates $|\phi>$ and has the properties of a good BRST charge. It is nilpotent (in 26 dimensions). It is remarkable that its anticommutation relation with the b ghost field is:

$$\{Q_{BRST}, b(z)\} = T^{Tot}(z) \tag{1}$$

where $T^{Tot}(z)$ is the geometrically normal ordered total energy-momentum tensor. This allows us to claim the BRST invariance of $|\phi>$ ($|\chi>$ tensored with the state associated on the same grounds with the matter fields X^μ) in 26 dimensions. This does not depend on the particular choice of coordinate around P. This is a nontrivial feature of Q_{BRST} because Q_{BRST} is not at first sight conformally invariant: if you want T^{Tot} to transform as a honest quadratic differential, then, the combination $T^{(X)} + \frac{1}{2}T^{bc}$ which enters in the definition of Q_{BRST}, does not do it.

2 Dual basis

Let K be the canonical line bundle o
P be a generic point P$\in \Sigma$. Consider
(quadratic differentials) and K^{-1} (ve
According to the Weierstrass gap theo
in a local coordinate z on a disc D ar
bases for such spaces of the form:

$$S_{K^1} = (z^{K^1} + \sum_{m^2}$$

$$V_{L^2} = (z^{-L^2+1} +$$

where for later convenience all indice
$k+1$ and all indices with superindex 2
An immediate consequence is

$$\oint_C S_{K^1} V$$

$$\Rightarrow B^2_{K^1}$$

as we can see by pulling off the surfac
We can complete those bases to ob
and vector in the coordinate z by add
1) a basis of all quadratic differenti
extend holomorphically to the rest of
2) a basis of all vectors which are m
morphically to $\Sigma - D$
3) a basis for all holomorphic vecto
cally to $\Sigma - D$.
Respectively:

$$1)S_{K^2} = z^{K}$$

$$2)V_{K^1} = z^{-}$$

$$2)V_{K^1} = z^{K}$$

As a consequence of (2.1), the basis $\{S_K\}$ and $\{V_L\}$ are dual with the scalar product

$$< S_K|V_L > \stackrel{\mathrm{def}}{=} \oint_{C_P} S_K V_L = \delta_{K\,L} \tag{5}$$

where C_P is a contour on D with index +1 in respect of P[1]

To be more specific, are dual between them: (1) the vectors which extend holomorphically "off P" and the quadratic differentials which are holomorphic on D but do not extend and vice versa, and (2) the globally defined quadratic differentials and the vectors which are neither holomorphic on D nor on $\Sigma - D$. These later vectors correspond to the Teichmüller deformations and induce the moduli changes.

It is also straightforward that acting on vectors written as kets and quadratic differentials written as bras one has the relation

$$\sum_K |V_K >< S_K| = 1 \tag{6}$$

We can see this by explicitly descomposing these in Fourier modes:

$$\begin{aligned} \mathcal{B}_L &= < b|V_L >= \oint_{C_P} bV_L \\ \mathcal{C}_K &= < S_K|c >= \oint_{C_P} S_K c \end{aligned} \tag{7}$$

and summing

$$\begin{aligned} \sum_L \mathcal{B}_L S_L &= \sum_L < b|V_L >< S_L| = b \\ \sum_K \mathcal{C}_K V_K &= \sum_K |V_K >< S_K|c >= c \end{aligned} \tag{8}$$

The relation with the usual descomposition

$$b = \sum_n b_n z^{n-2}(dz)^2$$

[1]In general there is neither globally defined holomorphic vectors nor quadratic differentials that are neither holomorphic on D nor on $\Sigma - D$. The only exception is the sphere. We focus our attention mainly in the $g \geq 1$ case, but the usual sphere and torus cases can be treated on the same foot.

$$c = \sum_n$$

is

$$\begin{aligned} \mathcal{B}_{K^1} &= b_{K^1} \\ \mathcal{B}_{K^2} &= b_{K^2} \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{K^1} &= c_{-K} \\ \mathcal{C}_{K^2} &= c_{-K} \end{aligned}$$

We will use these decompositions
momentum tensor of them and the
case we have

$$\begin{aligned} \{\mathcal{C}_K, \mathcal{B}_L\} &= \\ \{\mathcal{C}_K, \mathcal{C}_L\} &= \end{aligned}$$

Recall now that the ghost part of the
g Riemann surface $|\chi >$ is defined as
operators [1] . Such a state $|\chi >$ ca
transformation of the ghost vacuum
$k = 3g - 3$ units. This shift is necess
violation by this quantity in genus g
state $|k >$ and it is defined by

$$\begin{aligned} c_n|k > &= 0 \quad n \\ b_n|k > &= 0 \quad n \\ < k|k > &= 0 \\ g(B) &= exp\{- \\ |\chi > &= g(B)| \end{aligned}$$

Note that

$$g(B)\, c_{-l}\, g(B)^{-1} \;=\; \mathcal{C}_l$$
$$g(B)\, b_l\, g(B)^{-1} \;=\; \mathcal{B}_l \qquad (13)$$

and then we see that the condition defining $|\chi>$:

$$\mathcal{B}_{K^2}|\chi> \;=\; 0 \qquad (14)$$
$$\mathcal{C}_{L^1}|\chi> \;=\; 0 \qquad (15)$$

is simply the generalization via Bogoliubov transformation of the condition which defines $|k>$.

We can also think of a Bogoliubov transformation as a different form of filling the negative energy states [1]. The creation operators are invariant under Bogoliubov transformations, and the annihilation operators transform into the annihilation operators of the Bogoliubov-transformed vacuum. These observations have general validity.

Finally note that the vectors verify the Lie algebra

$$[V_M, V_N] \;=\; \sum_L C^L_{M\,N} V_L \qquad (16)$$
$$C^L_{M\,N} \;=\; \oint_{C_P} [V_M, V_N] \qquad (17)$$

where $[,]$ is the Lie bracket

$$[f(z)\partial_z, g(z)\partial_z] = (f(z)\partial_z g(z) - g(z)\partial_z f(z))\partial_z \qquad (18)$$

and the structure constants of the form $C^{L^1}_{M^2\,N^2}$ vanish

$$C^{L^1}_{M^2\,N^2} = 0 \qquad (19)$$

3 Geometric normal ordering

We are now ready to introduce a natural normal ordering for anomalous dimension $h = 2, -1$ conformal fields. Recall first an elementary fact in quantum

field theory: the normal ordering is d
always place on the right the modes
the remainder (the creation modes).
the normal ordering results in the su
espectation values and finally of the
in which the divergences appear.

The most natural thing we can do is
generalize the usual normal ordering
in the ghost case, and $|0>_X$ in the
now as reference for the normal or
the Bogoliubov transformed vacuum.
properties of the Riemann surface we

Consequently we define a geometri
$\star\ \star$ as follows: the Bogoliubov-trans
placed to the right and the remainde
Bogoliubov transformation) are to be

In order to complete the paralelis
introduce a state $|-k>$ of ghost nu
and defined by

$$c_n|-k> \;\;$$
$$b_n|-k> \;\;$$

$$<-k|-k> \;\;$$
$$<-k|+k> \;\;$$

which verifies the property

$$<-k|\mathcal{C}_{K^2}$$

Using this state tensored with the mat
in expectation values on the surface ha
b insertions in $<\Theta|...|\phi>$ in order to
$<\Theta|$ is right-annihilated by all (ghos

It is now clear that our definition
Fourier modes of the operators is co

[2] $|\phi>$ is the state of the ghost system tens
fields.

the $< \Theta|...|\phi >$ expectation values, which probably contain divergencies. (In higher genus Riemann surfaces the divergent terms in vacuum espectation values are the same as in the $g = 0, 1$ cases. We are doing *local* field theory. But in addition there are finite terms which depends on the genus of the surface. It is necessary also to substract these finite "global" terms.) Thus

$$\star A(z)B(z)\star \stackrel{\text{def}}{=} \{A(z)B(w) - < \Theta|A(z)B(w)|\phi >\}_{w\to z} \tag{22}$$

The analog of the Wick theorem is valid and

$$< \Theta| \star O(z) \star |\phi >= 0 \tag{23}$$

for any geometrically normal ordered operator $\star O(z)\star$.

It is a nice exercice to calculate now the bc two point function by using the basis $\{S_K\}$ and $\{V_L\}$, the expansion (2.7), the definition of $|\phi >$ and(3.2).

4 Virasoro algebra

The energy-momentum tensor Fourier modes are

$$\begin{aligned} \mathcal{L}^{bc}_M &\stackrel{\text{def}}{=} \oint_{C_P} dz V_M(z) \star T^{bc} \star \\ \star T^{bc}\star &= \star(2\partial_z cb + c\partial_z b)\star \\ \mathcal{L}^{(X)}_M &\stackrel{\text{def}}{=} \oint_{C_P} dz V_M(z) \star T^{(X)} \star \\ \star T^{(X)}\star &= -\frac{1}{2} \star (\partial_z X \partial_z X)\star \end{aligned} \tag{24}$$

They are the direct generalization of the usual L_m via $g(B)$). Their algebra is

$$[\mathcal{L}^{(X)}_M, \mathcal{L}^{(X)}_N] = \sum_L C^L_{N\,M} \mathcal{L}^{(X)}_L + A^{(X)}_{M\,N} \tag{25}$$

$$\tag{26}$$

$$[\mathcal{L}^{bc}_M, \mathcal{L}^{bc}_N] = \sum_L C^L_{N\,M} \mathcal{L}^{(X)}_L + A^{bc}_{M\,N} \tag{27}$$

where

$$\begin{aligned} A^{(X)}_{M\,N} &= \oint_{C_P} dw \oint_{C_w} dzV \\ A^{bc}_{M\,N} &= \oint_{C_P} dw \oint_{C_w} dzV \\ &+ < \partial c \partial b >< bc \end{aligned}$$

(Here $< ... >=< \Theta|...|\phi >$).

Observe first that now

$$\mathcal{L}^{bc}_M$$

is a simple consequence of the definit
normal ordering and (2.16).

Calculating now the leading terms
have as expected

$$A^{(X)}_{M\,N}$$

and the conformal anomaly cancels i
fields.

5 BRST charge

The generalization of the BRST char

$$\begin{aligned} Q_{BRST} &\stackrel{\text{def}}{=} \star \sum_M (\mathcal{C}_M \mathcal{L} \\ &= \star \sum_M \mathcal{C}_M (\\ &= \oint_{C_P} dz \star c \end{aligned}$$

This charge verifies

$$\begin{aligned} \{Q_{BRST}, c(z)\} &= \star c(z \\ \{Q_{BRST}, b(z)\} &= \star T^T \end{aligned}$$

and, if the total conformal anomaly vanishes

$$Q^2_{BRST} = \frac{1}{2}\{Q, Q\} = 0 \tag{35}$$

As we have explained in section 1, this property, combined with (5.3), ensures Q_{BRST} is well defined when we change of coordinate patch.

$|\phi>$ verifies

$$Q_{BRST}|\phi >= 0 \tag{36}$$

6 Conclusion

We can conclude that the geometric normal ordering we have introduced in this paper is the natural extension of the usual normal ordering on the sphere. The generalization to more punctures seems to be straightforward, The two punctures case involves the Krichever-Novikov algebras [5]. The point is the different Hilbert spaces associated to each puncture.

The generalization to super-Riemann surfaces seems also straightforward because there are analogs of the Riemann-Roch [4] and Weierstrass gap [3] theorems.

Work is in progress on these subjects.

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